

Power sums triangle

A self-constructive Pascal's type triangle

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Sums of the form $1^k + 2^k + \dots + n^k$, have been studied for centuries, where n and k are positive integers. We introduce a self-constructive, Pascal-type triangle. This operates similarly to how Pascal's triangle handles expansions of the form $(a+b)^n$, offering an approach despite the existence of Newton's binomial theorem.

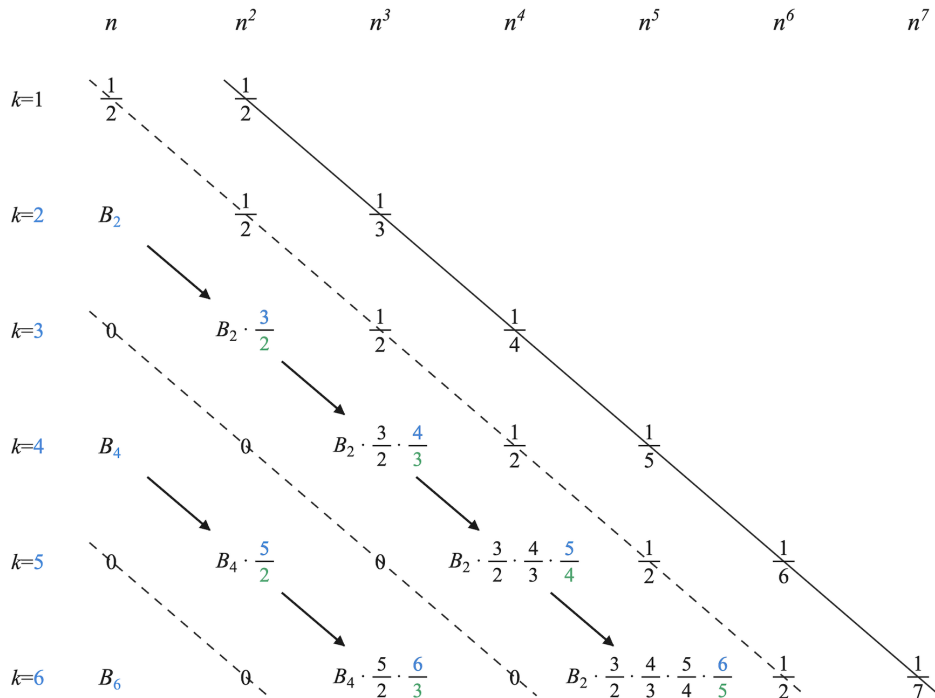
Self-Construction Principle. The construction of the triangle is governed by the following rules:

- **Initial Condition:** The first row starts with $\frac{1}{2}$ and $\frac{1}{2}$, representing the formula for the sum of the first n integers:

$$1 + 2 + \dots + n = \frac{n(n+1)}{2} = \frac{1}{2} \cdot n + \frac{1}{2} \cdot n^2$$

- **Rightmost Leading Diagonal:** The coefficients along this diagonal are equal $\frac{1}{k+1}$.
- **First Rightmost Dashed Diagonal:** The coefficients along this diagonal remain constant at $\frac{1}{2}$.
- **Row Initialization:** Subsequent rows start with B_k , where B_k are the *Bernoulli numbers*.
- **Recursive Step:** New row terms are built recursively by multiplying the previous diagonal term by $\frac{k}{p}$, where k is the row index and p is the power of n .

$B_k = 0$ for odd $k > 1$, so odd rows begin with 0 while even rows start with $B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}, B_6 = \frac{1}{42}, \dots$



Application. Example ($k = 4$). Calculate the sum $1^4 + 2^4 + \dots + n^4$.

Solution. We have $1^4 + 2^4 + \dots + n^4 = \frac{1}{5} \cdot n^5 + \frac{1}{2} \cdot n^4 + B_2 \cdot \frac{3}{2} \cdot \frac{4}{3} \cdot n^3 + 0 \cdot n^2 + B_4 \cdot n$. Thus

$$1^4 + 2^4 + \dots + n^4 = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30} = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}.$$

References

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