

Trapezoid law, Trapezoid theorem and their applications

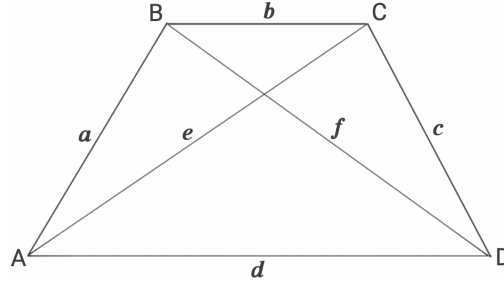
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Abstract

In this paper, we establish some novel results. In particular: a generalization of parallelogram law (we call it **Trapezoid law**), and a stronger result (diagonal formulas for expressing each diagonal of a trapezoid separately using its side lengths, we call it a **Trapezoid theorem**). We also investigate their applications.

Introduction

Consider the following trapezoid with side lengths a, b, c, d and diagonals e, f .



Trapezoid law (generalization of parallelogram's law). We have

$$a^2 + c^2 = e^2 + f^2 - 2bd.$$

Trapezoid theorem (stronger result). We have

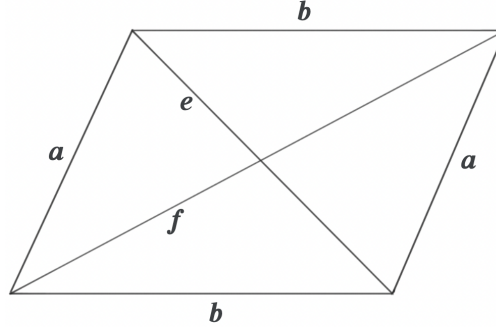
$$e^2 = bd + \frac{c^2b - a^2d}{b - d},$$

$$f^2 = bd + \frac{a^2b - c^2d}{b - d}.$$

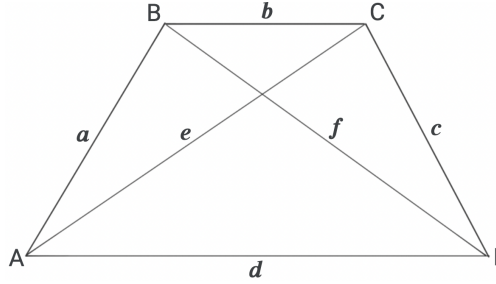
Remark 1. Note that Trapezoid law is the generalization of Parallelogram law.

Recall that the Parallelogram law states that

$$2a^2 + 2b^2 = e^2 + f^2.$$



The trapezoid below (see the figure) becomes a parallelogram, if $b = d$. In that case, we get $a = c$.



Trapezoid law states that

$$a^2 + c^2 = e^2 + f^2 - 2bd.$$

If we take $b = d$ and $a = c$, then we get

$$2a^2 = e^2 + f^2 - 2b^2.$$

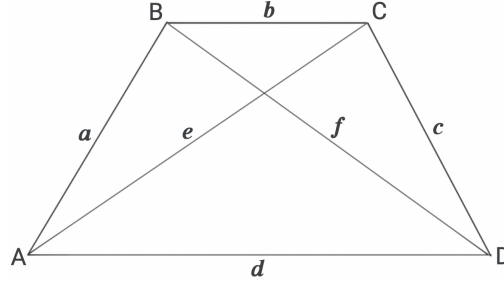
Therefore, we get the Parallelogram law.

$$2a^2 + 2b^2 = e^2 + f^2.$$

Remark 2. In the literature, there are 2 different definitions of a trapezoid. The first definition states that *trapezoid is a quadrilateral that has **exactly** one pair of parallel sides*, while the second definition states that *trapezoid is a quadrilateral that has **at least** one pair of parallel sides*. Trapezoid law shows that a parallelogram can actually be considered as a particular case of a trapezoid, therefore it looks like that the second definition is more appropriate as a definition of a trapezoid.

Main results and proofs

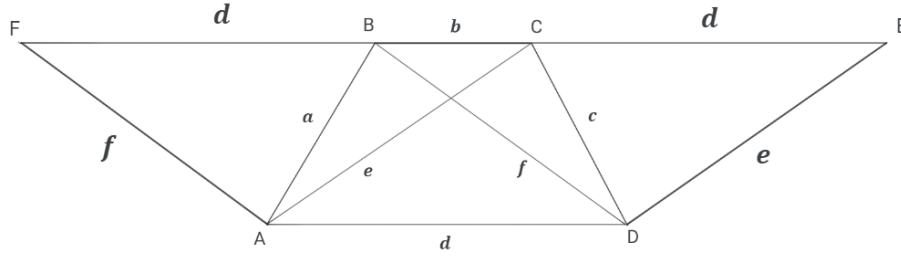
Consider the following trapezoid with side lengths a, b, c, d and diagonals e, f .



Trapezoid law (generalization of parallelogram's law). We have

$$a^2 + c^2 = e^2 + f^2 - 2bd.$$

Proof. Let us start by drawing AF parallel to BD and DE parallel to AC.



Using Stewart's theorem for triangles AFC and DBE , we get

$$a^2 = \frac{e^2 \cdot d + f^2 \cdot b}{d + b} - d \cdot b,$$

$$c^2 = \frac{e^2 \cdot b + f^2 \cdot d}{d + b} - d \cdot b.$$

Summing up the last two equations, we get

$$a^2 + c^2 = e^2 + f^2 - 2bd.$$

This ends the proof.

Trapezoid theorem (stronger result). We have

$$e^2 = bd + \frac{c^2b - a^2d}{b - d},$$

$$f^2 = bd + \frac{a^2b - c^2d}{b - d}.$$

Proof. From the equations of Stewart's theorem on previous page for triangles AFC and DBE , we get

$$f^2 = \frac{(d+b)(db+c^2) - e^2b}{d} = \frac{(d+b)(db+a^2) - e^2d}{b}.$$

Thus, it follows that

$$\frac{(d+b)(db+c^2) - e^2b}{d} - \frac{(d+b)(db+a^2) - e^2d}{b} = 0.$$

Therefore

$$\frac{(d+b)(db+c^2)}{d} - \frac{(d+b)(db+a^2)}{b} = \frac{e^2b}{d} - \frac{e^2d}{b}.$$

We get

$$\frac{(d+b)(db+c^2)}{d} - \frac{(d+b)(db+a^2)}{b} = \frac{e^2(b^2-d^2)}{db}.$$

Hence

$$b(d+b)(db+c^2) - d(d+b)(db+a^2) = e^2(b-d)(b+d).$$

Let us cancel $d+b$, then

$$b(db+c^2) - d(db+a^2) = e^2(b-d).$$

Therefore

$$e^2 = \frac{b(db+c^2) - d(db+a^2)}{b-d}.$$

Thus

$$e^2 = \frac{bd(b-d) - (a^2d - c^2b)}{b-d} = bd + \frac{c^2b - a^2d}{b-d}.$$

So, we get

$$e^2 = bd + \frac{c^2b - a^2d}{b-d}.$$

Now, we do the same with for f

$$e^2 = \frac{(d+b)(db+a^2) - f^2b}{d} = \frac{(d+b)(db+c^2) - f^2d}{b}.$$

We obtain

$$\frac{(d+b)(db+a^2) - f^2b}{d} - \frac{(d+b)(db+c^2) - f^2d}{b} = 0.$$

Hence

$$\frac{(d+b)(db+a^2)}{d} - \frac{(d+b)(db+c^2)}{b} = \frac{f^2b}{d} - \frac{f^2d}{b}.$$

Thus

$$\frac{(d+b)(db+a^2)}{d} - \frac{(d+b)(db+c^2)}{b} = \frac{f^2(b^2-d^2)}{db}.$$

We get

$$b(d+b)(db+a^2) - d(d+b)(db+c^2) = f^2(b-d)(b+d).$$

Let us cancel $d+b$

$$b(db+a^2) - d(db+c^2) = f^2(b-d).$$

Therefore

$$f^2 = \frac{b(db+a^2) - d(db+c^2)}{b-d}.$$

Thus, it follows that

$$f^2 = \frac{bd(b-d) + a^2b - c^2d}{b-d} = bd + \frac{a^2b - c^2d}{b-d}.$$

So, we get

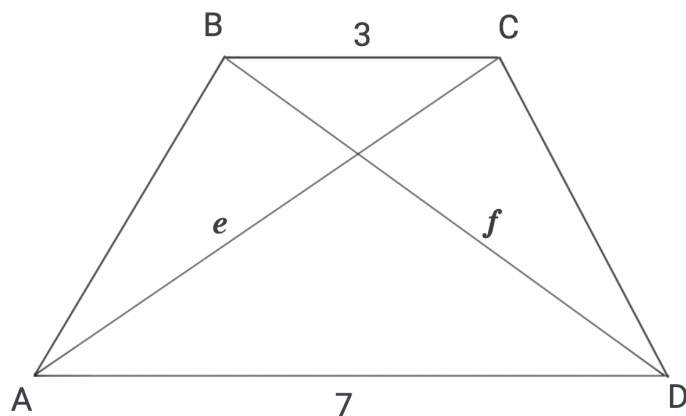
$$f^2 = bd + \frac{a^2b - c^2d}{b-d}.$$

This ends the proof.

Applications

Application 1 (Sedrakyan, The Stair-Step Approach in Mathematics). Consider a trapezoid with bases equal to 3 and 7. Given that the sum of the squares of its diagonals is equal to 100. Find the sum of the squares of its legs.

Solution. Let us consider the following trapezoid (see figure below).



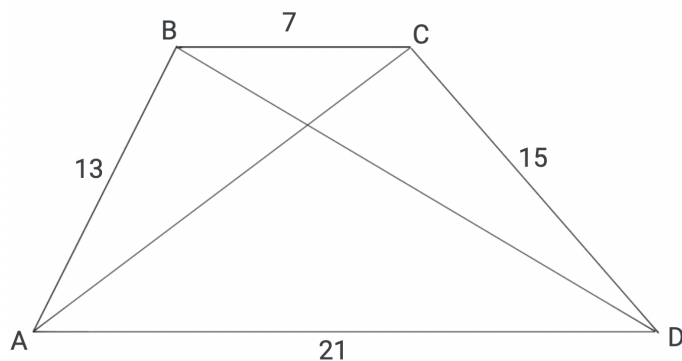
By trapezoid law, we have that

$$AB^2 + CD^2 = e^2 + f^2 - 2 \cdot BC \cdot AD.$$

We get

$$AB^2 + CD^2 = 100 - 42 = 58.$$

Application 2. Let ABCD be a trapezoid, where $AB = 13$, $BC = 7$, $CD = 15$, $AD = 21$ (see the figure). What is the length of diagonals AC and BD?



Solution. To find diagonal AC we use the Trapezoid theorem

$$AC^2 = 7 \cdot 21 + \frac{15^2 \cdot 7 - 13^2 \cdot 21}{7 - 21} = 288.$$

So, we get that

$$AC^2 = 288, \text{ so } AC = 12\sqrt{2}.$$

To find diagonal BD we use the Trapezoid theorem

$$BD^2 = 7 \cdot 21 + \frac{13^2 \cdot 7 - 15^2 \cdot 21}{7 - 21} = 400.$$

Thus, it follows that

$$BD^2 = 400, \text{ so } BD = 20.$$

References

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