

A Unified Distance-Based Area Formula for Convex Polygons

Hayk Sedrakyan, Nairi Sedrakyan, Christophe Pinarbasi

Abstract

This work fills a ~2000-year-old gap and is the structural “missing link”: Heron’s formula dates back to ~60 AD [1] and Bretschneider’s formula to 1842 [2, 3], yet no such unified formula existed for polygons with more than 4 sides. This work shows that these two formulas—which looked completely different for centuries—are particular cases of a single general formula. The remarkable thing is not that it uses only side lengths and certain diagonal lengths, but rather that one single unified formula computes the area of any convex n -gon for any number of sides $n \geq 3$ (which is the key novelty), previously thought impossible for $n \geq 5$. This novel formula is overcoming the “impossibility” of rigidity for $n \geq 5$.

Introduction

In computer graphics, geographic information systems, and structural engineering, calculating the area of irregular polygons usually requires breaking them down into a mesh of triangles or using Cartesian coordinates. In many real-world scenarios (flexible structures, robotics, distance-only sensors) coordinates are genuinely unavailable, making this formula not just a niche alternative but the only clean option. This newly discovered formula allows for the area calculation of a flexible or moving convex structure using only distance constraints (sides and specific internal diagonal lengths), bypassing the need to calculate coordinate positions or interior angles. The formula is expressed via a beautifully alternating sum of squared side lengths and a geometrically constructed point B_2 , which elegantly bypasses the structural flexibility of the polygon by introducing a single reference point relative to the diagonal A_1A_3 . The distance A_2B_2 accounts for the degrees of freedom of the polygon without requiring trigonometry, coordinates, or triangulation.

Theorem [Sedrakyan’s area formula]. The area of a convex n -gon $A_1A_2\dots A_n$ is:

$$\frac{1}{4} \sqrt{4A_1A_3^2 \cdot A_2B_2^2 - \left(a_1^2 - a_2^2 + \dots + (-1)^{n-1}a_n^2\right)^2}.$$

where $A_iA_{i+1} = a_i$ (indices mod n) and point B_2 is described in the Theorem provided in *the main result*.

Remark. This formula has been presented at international conferences as *Sedrakyan’s area formula* [4]. The formula was initially realized by *Hayk Sedrakyan*, experimentally verified with *Christophe Pinarbasi*, and formally proved by *Nairi Sedrakyan* and *Hayk Sedrakyan*. The author thanks Christophe Pinarbasi for stimulating discussions. Having recognized the result but lacking a formal proof at the time, the authors have, since 2022, shared this topic with several people; however, none of them was able to prove the result. Nevertheless, the authors would like to thank them for their efforts.

Particular cases (Heron’s and Bretschneider’s formulas)

Theorem ($n = 3$, Heron’s formula). The area A_T of a triangle $A_1A_2A_3$ with side lengths a, b, c is (in the section *main results and proofs* we show that for a triangle point B_2 **coincides with** vertex A_1):

$$A_T = \frac{1}{4} \sqrt{4 \cdot A_1A_3^2 \cdot A_2B_2^2 - (a_1^2 - a_2^2 + a_3^2)^2} = \frac{1}{4} \sqrt{4a^2c^2 - (a^2 - b^2 + c^2)^2}.$$

In the next section we show that this formula of A_T is actually Heron’s formula.

Theorem ($n = 4$, Bretschneider’s formula). The area A_Q of a quadrilateral $A_1A_2A_3A_4$ with side lengths a, b, c, d and diagonals e, f is (in the section *main results and proofs* we show that for a quadrilateral point B_2 coincides with A_4):

$$A_Q = \frac{1}{4} \sqrt{4 \cdot A_1A_3^2 \cdot A_2B_2^2 - (a_1^2 - a_2^2 + a_3^2 - a_4^2)^2} = \frac{1}{4} \sqrt{4e^2f^2 - (a^2 - b^2 + c^2 - d^2)^2}.$$

In the next section we show that this formula of A_Q is actually Bretschneider’s formula.

Remark. In applied sciences, we often encounter situations where we can measure distances, but we have absolutely no way to map out an (x, y) coordinate grid. For example:

Distance-Only Sensor Networks: Imagine a swarm of drones or localized robotic nodes. They can ping each other to measure their exact mutual distances (the side lengths and select diagonals of an n -gon), but they do not know their orientation in global space. They have no coordinate origin.

Flexible Structural Mechanics: Imagine an engineering framework or a architectural mesh under stress. The beams (sides) are shifting and flexing. In these scenarios, you cannot use the coordinate-based formula because you have no coordinates. This area formula provides the only direct mathematical pathway to compute or track the changing area purely out of a distance matrix.

Proof of equivalence for the cases $n = 3$ and $n = 4$

The case $n = 3$ (Heron's formula). Let us prove that the formula of A_T is actually Heron's formula. By difference of squares formula, we have

$$\begin{aligned} A_T &= \frac{1}{4} \sqrt{4a^2c^2 - (a^2 - b^2 + c^2)^2} = \frac{1}{4} \sqrt{(2ac - (a^2 - b^2 + c^2))(2ac + (a^2 - b^2 + c^2))} = \\ &= \frac{1}{4} \sqrt{(b^2 - (a^2 - 2ac + c^2))((a^2 + 2ac + c^2) - b^2)} = \frac{1}{4} \sqrt{(b^2 - (a - c)^2)((a + c)^2 - b^2)} = \\ &= \frac{1}{4} \sqrt{(b - (a - c))(b + (a - c))(a + c - b)(a + c + b)}. \end{aligned}$$

Thus, it follows that

$$A_T = \frac{1}{4} \sqrt{4a^2c^2 - (a^2 - b^2 + c^2)^2} = \frac{\sqrt{(b - a + c)(b + a - c)(a + c - b)(a + b + c)}}{4}.$$

The case $n = 4$ (Bretschneider's formula). Let us prove that the formula for A_Q is actually Bretschneider's formula. To find the area A_Q of any quadrilateral Bretschneider's formula states:

$$A_Q = \sqrt{(s - a)(s - b)(s - c)(s - d) - abcd \cos^2 \frac{\alpha + \gamma}{2}}, \quad s = \frac{a + b + c + d}{2},$$

where α, γ are opposite angles. Let θ be the angle between diagonals e and f . Let us prove that

$$\frac{1}{4} \sqrt{4e^2f^2 - (a^2 - b^2 + c^2 - d^2)^2} = \frac{1}{2} ef \sin \theta = \sqrt{(s - a)(s - b)(s - c)(s - d) - abcd \cos^2 \frac{\alpha + \gamma}{2}}.$$

Step 1: Our formula equals $\frac{1}{2} ef \sin \theta$. Place the quadrilateral in the plane with A_1 at the origin and let $\vec{e} = \overrightarrow{A_1A_3}$, $\vec{u} = \overrightarrow{A_1A_2}$, $\vec{v} = \overrightarrow{A_1A_4}$. We get

$$\begin{aligned} a^2 - b^2 + c^2 - d^2 &= |\vec{u}|^2 - |\vec{u} - \vec{e}|^2 + |\vec{v} - \vec{e}|^2 - |\vec{v}|^2 = 2(\vec{u} - \vec{v}) \cdot \vec{e} = 2ef \cos \theta. \text{ Thus} \\ \frac{1}{4} \sqrt{4e^2f^2 - (a^2 - b^2 + c^2 - d^2)^2} &= \frac{1}{4} \sqrt{4e^2f^2 - 4e^2f^2 \cos^2 \theta} = \frac{1}{4} \cdot 2ef \sin \theta = \frac{1}{2} ef \sin \theta. \end{aligned}$$

Step 2: Bretschneider's formula also equals $\frac{1}{2} ef \sin \theta$. Splitting the quadrilateral along diagonal e into two triangles gives:

$$A_Q = \frac{1}{2} ab \sin \alpha + \frac{1}{2} cd \sin \gamma,$$

and by the law of cosines $e^2 = a^2 + b^2 - 2ab \cos \alpha = c^2 + d^2 - 2cd \cos \gamma$. Using the identity $2 \sin \alpha \sin \gamma = \cos(\alpha - \gamma) - \cos(\alpha + \gamma)$ and expanding $(s - a)(s - b)(s - c)(s - d)$, one verifies by direct algebra that:

$$16(s - a)(s - b)(s - c)(s - d) - 16abcd \cos^2 \frac{\alpha + \gamma}{2} = 4e^2f^2 \sin^2 \theta,$$

and hence Bretschneider's formula gives $A_Q = \frac{1}{2} ef \sin \theta$.

Conclusion. Both formulas equal $\frac{1}{2} ef \sin \theta$, so they are equivalent.

The main result

Theorem [Sedrakyan's area formula]. The area of a convex n -gon $A_1A_2\dots A_n$ is:

$$\frac{1}{4}\sqrt{4A_1A_3^2 \cdot A_2B_2^2 - \left(a_1^2 - a_2^2 + \dots + (-1)^{n-1}a_n^2\right)^2}.$$

where $A_iA_{i+1} = a_i$ (indices mod n), and $B_2 = \ell_2 \cap \ell'_2$, lines ℓ_2 and ℓ'_2 are defined:

- ℓ_2 : the line perpendicular to A_1A_3 passing through the point $(x_0, 0)$, where

$$x_0 = \frac{a^2 - a_3^2 + a_4^2 - \dots + (-1)^n a_n^2}{2a}$$

and $a = A_1A_3$, we take $A_1 = (0, 0)$, $A_3 = (a, 0)$.

- ℓ'_2 : the line parallel to A_1A_3 passing through the point $(0, y_0)$, where

$$y_0 = \frac{2((A_1A_2A_3) - (A_1A_2\dots A_n))}{a}$$

lying on the side of A_1A_3 opposite to A_2 , and $a = A_1A_3$.

Important (constructing the point B_2). Looking at the form of y_0 in the previous equation, it initially appears that the area of $(A_1A_2\dots A_n)$ is required to construct line ℓ'_2 and, consequently, to locate point B_2 . This seems to introduce a circular argument, implying that we must already know the area of $(A_1A_2\dots A_n)$ to find it. However, in Remark 1 of the next section, we demonstrate that the value of this area is not actually needed to construct ℓ'_2 and determine the location of B_2 .

The meaning of the formula

At first glance, the introduction of the point B_2 may seem somewhat unusual. However, consider the analogous concept of the center of mass of a three-dimensional body. The center of mass is a single point that represents the distribution of mass throughout the body, and it is well known that this point may lie outside the body itself. In two dimensions, instead of volume, we consider area. The concept of volume in three dimensions corresponds to the concept of area in two dimensions. In a similar way, the point B_2 can be viewed as a representative point associated with the area distribution of the figure.

The proof of the main result

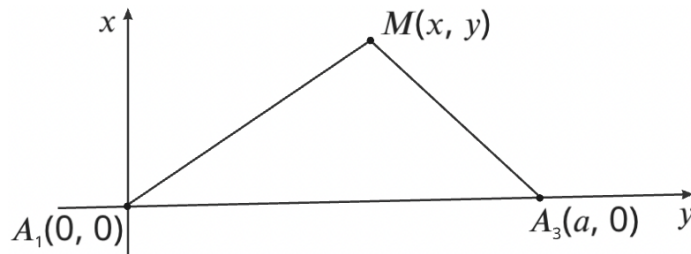
Let us denote the area of a convex n -gon $A_1A_2\dots A_n$ by $(A_1A_2\dots A_n)$.

Let $A_1A_2 = a_1$, $A_2A_3 = a_2, \dots$, $A_{n-1}A_n = a_{n-1}$, $A_nA_1 = a_n$.

Problem 1. Find the locus of all points M in the plane such that

$$MA_3^2 - MA_1^2 = a_3^2 - a_4^2 + \dots + (-1)^{n-1}a_n^2.$$

Solution of Problem 1. Choose a coordinate system as shown in the figure.



We have

$$\begin{aligned} MA_3^2 - MA_1^2 &= a_3^2 - a_4^2 + \dots + (-1)^{n-1} a_n^2. \text{ Thus} \\ (x-a)^2 + y^2 - x^2 - y^2 &= a_3^2 - a_4^2 + \dots + (-1)^{n-1} a_n^2. \\ x &= \frac{a^2 - a_3^2 + \dots + (-1)^n a_n^2}{2a}. \end{aligned}$$

Let us prove that the locus of all points M in the plane is line l_2 which passes through the point

$$\left(\frac{a^2 - a_3^2 + \dots + (-1)^n a_n^2}{2a}, 0 \right)$$

and is perpendicular to line A_1A_3 .

If $n = 3$, then $a = a_3$ and line l_2 passes through the point $A_1(0, 0)$.

If $n = 4$, then line l_2 is passing through point A_4 , as

$$A_4A_3^2 - A_4A_1^2 = a_3^2 - a_4^2.$$

Problem 2. Find the locus of all points M in the plane such that $(MA_1A_2A_3) = (A_1A_2\dots A_n)$ and that points M and A_2 are not on the same side of line A_1A_3 .

Solution of Problem 2. Choose a coordinate system as shown in the solution to problem 1. Let $M(x, y)$, where $y \leq 0$, we have

$$\begin{aligned} \frac{(-y) \cdot a}{2} &= (A_1A_2\dots A_n) - (A_1A_2A_3). \text{ Thus} \\ y &= \frac{2\left((A_1A_2\dots A_n) - (A_1A_2A_3)\right)}{(-a)} = \frac{2\left((A_1A_2A_3) - (A_1A_2\dots A_n)\right)}{a}. \end{aligned}$$

Let us prove that the locus of all points M is line l'_2 passing through point

$$\left(0, \frac{2\left((A_1A_2A_3) - (A_1A_2\dots A_n)\right)}{a} \right)$$

and is perpendicular to line l_2 .

If $n = 3$, then line l'_2 coincides with line A_1A_3 , as

$$\frac{2\left((A_1A_2A_3) - (A_1A_2\dots A_n)\right)}{a} = 0.$$

If $n = 4$, then line l'_2 passes through point A_4 , as

$$\frac{2\left((A_1A_2A_3) - (A_1A_2A_3A_4)\right)}{a} = -\frac{2(A_1A_4A_3)}{a}.$$

Let us denote by B_2 the intersection point of lines l_2 and l'_2 , in a similar way denote by B_i the intersection point of lines l_i and l'_i , where $i = 1, \dots, n$.

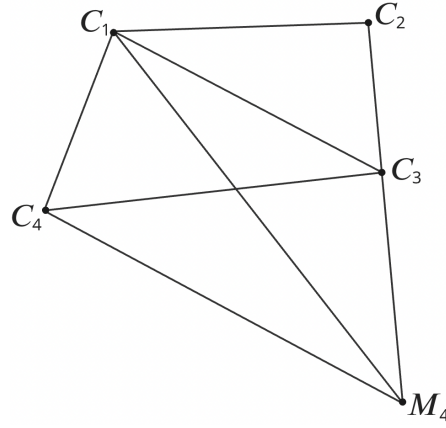
If $n \geq 5$, then let us start with the following remark.

Remark 1. Let a convex n -gon be given. Construct point M_n on line C_2C_3 such that

$$(M_nC_1C_2) = (C_1C_2\dots C_n).$$

Let us construct point M_n by mathematical induction.

The base case. If $n = 4$, then from point C_4 let us draw a line parallel to line C_1C_3 . Denote by M_4 the intersection point of that line and ray C_2C_3 (see figure).



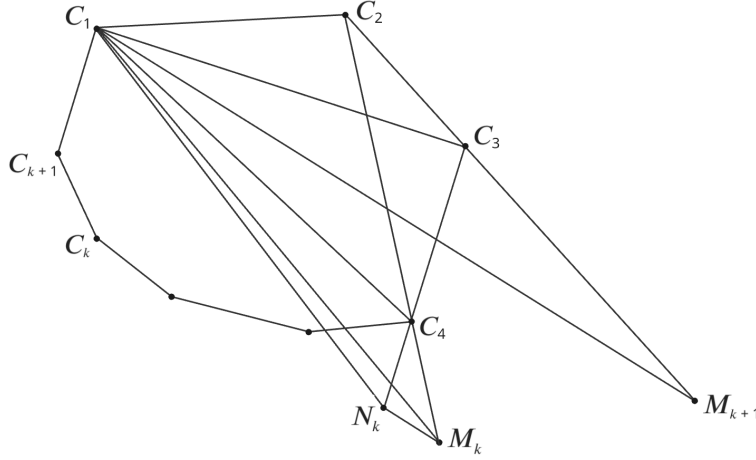
Indeed, we have

$$(M_4C_1C_2) = (C_1C_2C_3) + (M_4C_1C_3) = (C_1C_2C_3) + (C_4C_1C_3) = (C_1C_2C_3C_4).$$

The inductive hypothesis. Assume that for $n = k$ the point M_k is already constructed, then let us show how point M_{k+1} is constructed. Consider the convex polygon $C_1C_2C_4...C_{k+1}$. According to the inductive hypothesis on ray C_2C_4 we can construct point M_k such that

$$(M_kC_1C_2) = (C_1C_2C_4...C_{k+1}).$$

Let us draw a line from point M_k parallel to line C_1C_4 (see figure). Denote by N_k the intersection point of that line and line C_3C_4 .



We have

$$(N_kC_1C_4) = (M_kC_1C_4).$$

Thus

$$(N_kC_1C_2C_4) = (M_kC_1C_4) + (C_1C_2C_4) = (M_kC_1C_2) = (C_1C_2C_4...C_{k+1}).$$

Thus, we get that

$$(N_kC_1C_2C_3) = (C_1C_2C_3...C_{k+1}). \quad (1)$$

It is left for quadrilateral $C_1C_2C_3N_k$ on ray C_2C_3 to construct point M_{k+1} , such that

$$(M_{k+1}C_1C_2) = (C_1C_2C_3N_k) \quad (2)$$

From (1) and (2), we get

$$(M_{k+1}C_1C_2) = (C_1C_2C_3...C_{k+1}).$$

This ends the proof for $n = k + 1$. Thus, this ends the proof for any $n \geq 4$.

From (1), (2), (3), we get that

$$\begin{aligned} \frac{1}{2}A_1A_3 \cdot A'_2B_2 &= \sqrt{4A_1A_3^2 \cdot A_2B_2^2 - \left(a_1^2 - a_2^2 + a_3^2 - \dots + (-1)^{n-1}a_n^2\right)^2}. \text{ Thus} \\ (A_1A_2\dots A_n) &= \frac{1}{4}\sqrt{4A_1A_3^2 \cdot A_2B_2^2 - \left(a_1^2 - a_2^2 + a_3^2 - \dots + (-1)^{n-1}a_n^2\right)^2}. \end{aligned} \quad (4)$$

This ends the proof of the theorem.

Conclusion. Triangles ($n = 3$) are inherently rigid because fixing their side lengths completely locks their interior angles and area into place. Quadrilaterals, pentagons, and higher-order polygons ($n \geq 4$) are flexible, meaning their sides can deform and flex like a mechanical linkage, constantly changing their angles and overall area. This fundamental fluidness is why a unified, side-length-based area formula was long considered impossible for shapes with five or more sides. Conversely, in geometry, you cannot find the area of an irregular n -gon purely from its sides because the shape can bend. By capturing that exact architectural fluidness inside a single point B_2 and an alternating sum of squared sides, this area formula accomplishes something structurally revolutionary that did not have an existing. It proves that the chaotic flexibility of any arbitrary polygon can be structurally bound and unified under a single algebraic law, opening the door to solving distance-only constraints that were previously considered mathematically impossible.

Example showing how the formula works

Example. Let us find the area of a regular hexagon $A_1A_2\dots A_6$, where $A_1A_2 = 1$.

Solution. Line l_2 is determined by all points M satisfying the following equation

$$MA_3^2 - MA_1^2 = a_3^2 - a_1^2 + a_5^2 - a_6^2 = 0.$$

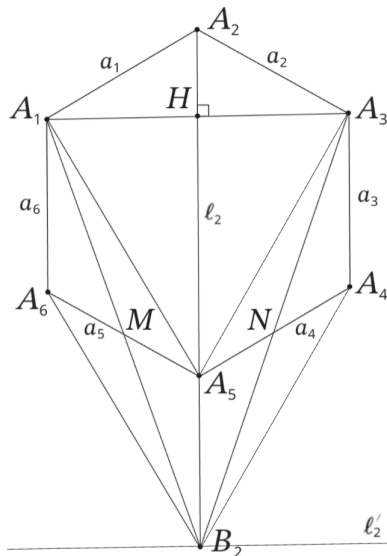
So, it will be the perpendicular bisector of line segment A_1A_3 (see figure).

Now, let us construct line l'_2 and find the intersection point B_2 of lines l_2 and l'_2 . Using the theorem we have that l'_2 is parallel to A_1A_3 and is passing through the point $(0, y_0)$, where

$$y_0 = \frac{2\left((A_1A_2A_3) - (A_1A_2A_3A_4A_5A_6)\right)}{A_1A_3} = \frac{-2(A_1A_3A_4A_5A_6)}{A_1A_3}.$$

Notice that $y_0 < 0$, so $-y_0 > 0$. We get

$$(A_1A_3A_4A_5A_6) = \frac{(-y_0) \cdot A_1A_3}{2} = \frac{B_2H \cdot A_1A_3}{2} = (B_2A_1A_3).$$



Therefore, point B_2 is a point on line l_2 , such that

$$(B_2A_1A_3) = (A_1A_3A_4A_5A_6).$$

When $A_1A_5 \parallel A_6B_2$ and $A_3A_5 \parallel A_4B_2$, then $(A_1MA_6) = (B_2MA_5)$ and $(A_3NA_4) = (B_2NA_5)$. Hence

$$(B_2A_1A_3) = (A_1A_3A_4A_5A_6),$$

as $A_1A_3NA_5M$ is in common for triangle $B_2A_1A_3$ and pentagon $A_1A_3A_4A_5A_6$. We obtain

$$A_2B_2 = A_2A_5 + A_5B_2 = 2 + A_1A_6 = 3, \quad A_1A_3 = \sqrt{3}.$$

Applying the area formula (note that $a_1 = a_2 = a_3 = a_4 = a_5 = a_6$), we get

$$(A_1A_2A_3A_4A_5A_6) = \frac{1}{4} \sqrt{4 \cdot 3 \cdot 9 - (a_1^2 - a_2^2 + a_3^2 - a_4^2 + a_5^2 - a_6^2)^2} = \frac{1}{4} \sqrt{4 \cdot 3 \cdot 9 - 0^2} = \frac{3\sqrt{3}}{2}.$$

Verification of the answer. A regular hexagon with side length 1 can be divided into six equilateral triangles of side length 1. The area of one equilateral triangle with side length 1 is

$$A_{\triangle} = \frac{\sqrt{3}}{4}.$$

Therefore, the area of the hexagon is

$$(A_1A_2A_3A_4A_5A_6) = 6 \cdot \frac{\sqrt{3}}{4} = \frac{3\sqrt{3}}{2}.$$

Applications

Application 1. A triangle has side lengths $\sqrt{3}$, $\sqrt{5}$, and $\sqrt{7}$. Find its area.

Application 2. Consider a trapezoid with bases of lengths 5 and 9. If the sum of the squares of its diagonals is 196, find the sum of the squares of its legs.

Application 3. The area of triangle with sides a , b , c is equal to the area of triangle with sides a , b , d , where $d \neq c$, and that $a^2 + b^2 = 16$, find $c^2 + d^2$.

Application 4. Each of two non-congruent triangles has two sides of lengths 5 and 7, and the triangles have equal circumradii. Find the product of the lengths of their third sides.

Application 5. Let a_1 , a_2 , a_3 be the side lengths of a triangle, prove that

- 1). $2a_1^2a_2^2 + 2a_2^2a_3^2 + 2a_1^2a_3^2 > a_1^4 + a_2^4 + a_3^4$.
- 2). $a_1^2a_2 + a_2^2a_1 + a_2^2a_3 + a_3^2a_2 + a_1^2a_3 + a_3^2a_1 > a_1^3 + a_2^3 + a_3^3$.

Application 6. Let $ABCD$ be a convex quadrilateral, such that $AB = a$, $BC = b$, $CD = c$, $AD = d$, $AC = e$, $BD = f$. Prove that

$$|a^2 - b^2 + c^2 - d^2| < 2ef.$$

Application 7. Let $A_1A_2...A_{2k}$ be a convex polygon such that $A_1A_3 = A_2A_4 = ... = A_{2k-2}A_{2k} = A_{2k-1}A_1 = A_{2k}A_2$. Prove that

$$A_1B_1 = A_2B_2 = ... = A_{2k}B_{2k}.$$

Solutions of applications

Application 1. A triangle has side lengths $\sqrt{3}$, $\sqrt{5}$, and $\sqrt{7}$. Find its area.

Solution. According to this area formula, we have

$$Area = \frac{1}{4} \sqrt{4 \cdot (\sqrt{3})^2 \cdot (\sqrt{7})^2 - \left((\sqrt{3})^2 - (\sqrt{5})^2 + (\sqrt{7})^2 \right)^2} = \frac{1}{4} \sqrt{4 \cdot 3 \cdot 7 - (3 - 5 + 7)^2} = \frac{\sqrt{59}}{4}.$$

Application 2. Consider a trapezoid with bases of lengths 5 and 9. If the sum of the squares of its diagonals is 196, find the sum of the squares of its legs.

Solution. Let the diagonals be e and f (see the figure). Given that $e^2 + f^2 = 196$. It is a known result that if $e^2 + f^2 = (a + b)^2$, where a and b are the bases then this condition is equivalent to the condition that the diagonals of the trapezoid must be perpendicular to each other. In this case the bases are 5 and 9, given that $e^2 + f^2 = 196 = (5 + 9)^2$. This means that the diagonals of this trapezoid must be perpendicular to each other. Then, we have

$$\text{Area of trapezoid} = \frac{e \cdot f}{2}.$$

On the other hand (by the considered area formula), we have

$$\text{Area of trapezoid} = \frac{1}{4} \sqrt{4e^2f^2 - (a^2 - 5^2 + c^2 - 9^2)^2}.$$

Thus, we get

$$\frac{ef}{2} = \frac{1}{4} \sqrt{4e^2f^2 - (a^2 - 5^2 + c^2 - 9^2)^2}.$$

Squaring both side of this equation, we get

$$\frac{e^2f^2}{4} = \frac{1}{16} (4e^2f^2 - (a^2 - 5^2 + c^2 - 9^2)^2).$$

This means that

$$(a^2 - 5^2 + c^2 - 9^2)^2 = 0.$$

So, we get

$$a^2 + c^2 = 5^2 + 9^2 = 106.$$

Application 3. The area of triangle with sides a , b , c is equal to the area of triangle with sides a , b , d , where $d \neq c$, and that $a^2 + b^2 = 16$, find $c^2 + d^2$.

Solution. As the areas of both triangles are equal, then we have

$$\frac{1}{4} \sqrt{4a^2b^2 - (a^2 + b^2 - c^2)^2} = \frac{1}{4} \sqrt{4a^2b^2 - (a^2 + b^2 - d^2)^2}.$$

Squaring both sides and canceling the expression $4a^2b^2$ on both side, we get

$$(a^2 + b^2 - c^2)^2 = (a^2 + b^2 - d^2)^2.$$

Therefore,

$$(a^2 + b^2 - c^2)^2 - (a^2 + b^2 - d^2)^2 = 0.$$

Note that this can be factored as follows:

$$\left(a^2 + b^2 - c^2 - (a^2 + b^2 - d^2) \right) \cdot \left(a^2 + b^2 - c^2 + (a^2 + b^2 - d^2) \right) = 0.$$

Simplifying, we get

$$(c^2 - d^2) \cdot (2(a^2 + b^2) - (c^2 + d^2)) = 0.$$

Given that $a^2 + b^2 = 16$, hence

$$(c^2 - d^2) \cdot (2 \cdot 16 - (c^2 + d^2)) = 0$$

Given also that $c \neq d$, thus $c^2 - d^2 \neq 0$. Therefore,

$$32 - (c^2 + d^2) = 0.$$

We obtain

$$c^2 + d^2 = 32.$$

Application 4. Each of two non-congruent triangles has two sides of lengths 5 and 7, and the triangles have equal circumradii. Find the product of the lengths of their third sides.

Solution of Application 3. Denote their areas by A_1 and A_2 , denote the length of their third sides by a and b , then by this area formula, we have

$$A_1 = \frac{1}{4} \sqrt{4 \cdot 5^2 \cdot 7^2 - (5^2 - a^2 + 7^2)^2},$$

$$A_2 = \frac{1}{4} \sqrt{4 \cdot 5^2 \cdot 7^2 - (5^2 - b^2 + 7^2)^2}.$$

Denote by R_1 and R_2 their circumradii, then by the law of sines, we have

$$A_1 = \frac{5 \cdot 7a}{4R_1},$$

$$A_2 = \frac{5 \cdot 7b}{4R_2}.$$

As $R_1 = R_2$, we get

$$\frac{5 \cdot 7a}{4A_1} = \frac{5 \cdot 7b}{4A_2}.$$

Thus, it follows that

$$\frac{a}{\sqrt{4 \cdot 5^2 \cdot 7^2 - (5^2 - a^2 + 7^2)^2}} = \frac{b}{\sqrt{4 \cdot 5^2 \cdot 7^2 - (5^2 - b^2 + 7^2)^2}}.$$

Squaring both sides, we get

$$a^2 \left(4 \cdot 5^2 \cdot 7^2 - (5^2 - b^2 + 7^2)^2 \right) = b^2 \left(4 \cdot 5^2 \cdot 7^2 - (5^2 - a^2 + 7^2)^2 \right).$$

Simplifying and using difference of squares formula, we get

$$(a^2 - b^2) \cdot 4 \cdot 5^2 \cdot 7^2 = \left(a(5^2 - b^2 + 7^2) - b(5^2 - a^2 + 7^2) \right) \left(a(5^2 - b^2 + 7^2) + b(5^2 - a^2 + 7^2) \right).$$

We obtain

$$(a^2 - b^2) \cdot 4 \cdot 5^2 \cdot 7^2 = (a - b)(5^2 + ab + 7^2)(a + b)(5^2 - ab + 7^2).$$

As triangles are different, then $a \neq b$. Canceling $a^2 - b^2$ and $(a - b)(a + b)$, we get

$$4 \cdot 5^2 \cdot 7^2 = (5^2 + 7^2)^2 - (ab)^2.$$

Hence

$$(ab)^2 = (5^2 + 7^2)^2 - 4 \cdot 5^2 \cdot 7^2 = (5^2 + 7^2 - 2 \cdot 5 \cdot 7)(5^2 + 7^2 + 2 \cdot 5 \cdot 7) = (25 + 49 - 70)(25 + 49 + 70) = 4 \cdot 144.$$

Therefore

$$ab = 2 \cdot 12 = 24.$$

Application 5. Let a_1, a_2, a_3 be the side lengths of a triangle, prove that

- 1). $2a_1^2a_2^2 + 2a_2^2a_3^2 + 2a_1^2a_3^2 > a_1^4 + a_2^4 + a_3^4.$
- 2). $a_1^2a_2 + a_2^2a_1 + a_2^2a_3 + a_3^2a_2 + a_1^2a_3 + a_3^2a_1 > a_1^3 + a_2^3 + a_3^3.$

Proof of Application 5. We have

$$4a_3^2a_1^2 - (a_1^2 - a_2^2 + a_3^2)^2 > 0,$$

and

$$(a_2 + a_3 - a_1)(a_2 + a_1 - a_3)(a_1 + a_3 - a_2) > 0.$$

This ends the proof of both inequalities of Problem 3.

If $n = 4$, then from (4) and from the fact that point B_2 coincides with point A_4 we get

$$(A_1A_2A_3A_4) = \frac{1}{4} \sqrt{4A_1A_3^2 \cdot A_2A_4^2 - (a_1^2 - a_2^2 + a_3^2 - a_4^2)^2}. \quad (5)$$

Problem 5. Let $ABCD$ be a convex quadrilateral, such that $AB = a, BC = b, CD = c, AD = d, AC = e, BD = f$. Prove that

$$|a^2 - b^2 + c^2 - d^2| < 2ef.$$

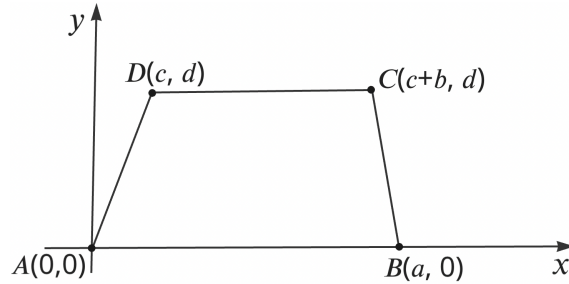
Proof of Application 5. The proof of Problem 4 follows from (5).

Application 7. Let $A_1A_2...A_{2k}$ be a convex polygon such that $A_1A_3 = A_2A_4 = ... = A_{2k-2}A_{2k} = A_{2k-1}A_1 = A_{2k}A_2$. Prove that

$$A_1B_1 = A_2B_2 = ... = A_{2k}B_{2k}.$$

Proof of Application 7. The proof of Problem 5 follows from the theorem.

Alternative proof of the lemma. Choose a coordinate system as in the solution to Problem 1.



We have

$$AC^2 + BD^2 - AD^2 - BC^2 = (c+b)^2 + d^2 + (c-a)^2 + d^2 - c^2 - d^2 - (c+b-a)^2 - d^2 = 2ab = 2 \cdot AB \cdot CD.$$

This ends the proof of the lemma.

References

- [1] Fabio Acerbi and Bernard Vitrac. *Heron of Alexandria: Metrica, Heron's Formula*. Serra, 2014.
- [2] Carl Anton Bretschneider. Untersuchung der trigonometrischen relationen des geradlinigen viereckes. *Archiv der Mathematik*, 2:225–261, 1842.
- [3] Julian Lowell Coolidge. A historically interesting formula for the area of a quadrilateral. *American Mathematical Monthly*, 46:345–347, 1939.
- [4] Hayk Sedrakyan and Nairi Sedrakyan. *Mathematics Competitions Must-Know Geometry: Practice and Theory Inspired by the Style of AMC and AIME Geometry (Unofficial Guide)*. 2023.