

Generalizations of Sedrakyan's inequality

Hayk Sedrakyan, Nairi Sedrakyan, Raphael Pinarbasi

Abstract

In 1997, *Nairi Sedrakyan* published an article titled *About the Applications of One Useful Inequality*, in which he proved the inequality for squares and for any k^{th} power. To the best of our knowledge, *Nairi Sedrakyan* was the first author to publish the inequality for any k^{th} power. He later republished his generalizations for the k^{th} power in the book *Algebraic Inequalities* (Springer). The inequality for squares was published by several authors and became popular under the name *Sedrakyan's inequality* (1997), or *Titu's lemma* (2003), referring to T. Andreescu's 2003 book *Mathematical Olympiad Treasures*. The inequality for squares is also known as *Engel's form* or *Bergström's inequality*.

Introduction

Generalization of Sedrakyan's inequality for the k^{th} power (n terms).

Let $a_1, a_2, \dots, a_n > 0$ and $b_1, b_2, \dots, b_n > 0$, $c_1, c_2, \dots, c_n > 0$, ..., $\gamma_1, \gamma_2, \dots, \gamma_n > 0$. Then

$$\underbrace{\frac{a_1^k}{b_1 c_1 \cdots \gamma_1}}_{k-1 \text{ terms}} + \underbrace{\frac{a_2^k}{b_2 c_2 \cdots \gamma_2}}_{k-1 \text{ terms}} + \cdots + \underbrace{\frac{a_n^k}{b_n c_n \cdots \gamma_n}}_{k-1 \text{ terms}} \geq \frac{(a_1 + a_2 + \cdots + a_n)^k}{(b_1 + b_2 + \cdots + b_n)(c_1 + c_2 + \cdots + c_n) \cdots (\gamma_1 + \gamma_2 + \cdots + \gamma_n)}.$$

In the section *Main Results and Proofs*, we provide the equality conditions.

Generalization of Sedrakyan's inequality for cubes (n terms).

Let $a_1, a_2, \dots, a_n > 0$ and $b_1, b_2, \dots, b_n > 0$, $c_1, c_2, \dots, c_n > 0$. Then

$$\frac{a_1^3}{b_1 c_1} + \frac{a_2^3}{b_2 c_2} + \frac{a_3^3}{b_3 c_3} + \cdots + \frac{a_n^3}{b_n c_n} \geq \frac{(a_1 + a_2 + \cdots + a_n)^3}{(b_1 + b_2 + \cdots + b_n)(c_1 + c_2 + \cdots + c_n)}.$$

In the section *Main Results and Proofs*, we provide the equality conditions.

Sedrakyan's inequality (Titu's lemma).

Let $a_1, a_2, \dots, a_n$ be real numbers and $b_1, b_2, \dots, b_n > 0$. Then

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \cdots + \frac{a_n^2}{b_n} \geq \frac{(a_1 + a_2 + \cdots + a_n)^2}{b_1 + b_2 + \cdots + b_n}.$$

Equality conditions

The first equality condition is the one for the k^{th} power and n terms.

Generalization for the k^{th} power (n terms), equality condition.

$$\left\{ \begin{array}{l} \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = \dots = \frac{a_n}{b_n}, \\ \frac{b_1}{c_1} = \frac{b_2}{c_2} = \frac{b_3}{c_3} = \dots = \frac{b_n}{c_n}, \\ \vdots \\ \frac{y_1}{\gamma_1} = \frac{y_2}{\gamma_2} = \frac{y_3}{\gamma_3} = \dots = \frac{y_n}{\gamma_n}. \end{array} \right.$$

The next one is the generalization for cubes with n terms.

Generalization for cubes (n terms), equality condition.

$$\left\{ \begin{array}{l} \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = \dots = \frac{a_{n-1}}{b_{n-1}}, \\ \frac{b_1}{c_1} = \frac{b_2}{c_2} = \frac{b_3}{c_3} = \dots = \frac{b_{n-1}}{c_{n-1}}. \end{array} \right.$$

The third one is Sedrakyan's inequality's (Titu's lemma) equality condition.

Sedrakyan's inequality (Titu's lemma), equality condition.

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = \dots = \frac{a_n}{b_n}.$$

Proofs of the initial inequality

In this section we provide the proof of Sedrakyan's inequality (Titu's lemma). In the next section (Main Results and Proofs) we provide the proofs of the generalizations and their corresponding equality conditions.

Proof. At first, we want to prove the following.

Sedrakyan's inequality (Titu's lemma).

Let $a_1, a_2, \dots, a_n$ be real numbers and $b_1, b_2, \dots, b_n > 0$. Then

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \dots + \frac{a_n^2}{b_n} \geq \frac{(a_1 + a_2 + \dots + a_n)^2}{b_1 + b_2 + \dots + b_n}.$$

In order to prove this inequality let us start with the case of two terms, that is: let $a > 0$ and $b > 0$, let us prove that

$$\frac{x^2}{a} + \frac{y^2}{b} \geq \frac{(x+y)^2}{a+b}.$$

Rewrite this expression using a common denominator:

$$\frac{x^2b + y^2a}{ab} \geq \frac{(x+y)^2}{a+b}.$$

Cross multiply to get:

$$x^2ba + y^2a^2 + x^2b^2 + y^2ab \geq x^2ab + 2xyab + y^2ab.$$

Then, cancel the common terms:

$$y^2a^2 + x^2b^2 \geq 2xyab.$$

Let us rewrite it this way:

$$y^2a^2 - 2xyab + x^2b^2 \geq 0.$$

Complete the square:

$$(ya - xb)^2 \geq 0.$$

As a square of any number is non-negative, this ends the proof of the initial statement.

Using the case of two terms, let us prove the same statement for three terms, that is:

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} \geq \frac{(x+y+z)^2}{a+b+c}.$$

Denote $x + y = A$ and $a + b = B$, then using the statement for two terms we get

$$\frac{(x + y)^2}{a + b} + \frac{z^2}{c} = \frac{A^2}{B} + \frac{z^2}{c} \geq \frac{(A + z)^2}{B + c} = \frac{((x + y) + z)^2}{(a + b) + c}.$$

Thus, it follows that

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} \geq \frac{(x + y)^2}{a + b} + \frac{z^2}{c} \geq \frac{((x + y) + z)^2}{(a + b) + c} = \frac{(x + y + z)^2}{a + b + c}.$$

This ends the proof of the statement for three terms.

In a similar way, we can prove the case of four terms, and have the same result with more terms like 5, 6, 7,... (up to n terms, where n is any positive integer).

This ends the proof of Sedrakyan's inequality (Titu's lemma).

Equality condition for two terms (second power). Let us prove the equality is possible (for two terms):

$$\frac{x^2}{a} + \frac{y^2}{b} = \frac{(x + y)^2}{a + b}.$$

Rewrite with common denominator

$$\frac{x^2b + y^2a}{ab} = \frac{(x + y)^2}{a + b}.$$

Cross multiply

$$x^2ba + y^2a^2 + x^2b^2 + y^2ab = x^2ab + 2xyab + y^2ab.$$

Cancel common terms

$$y^2a^2 + x^2b^2 = 2xyab.$$

$$(ya - xb)^2 = 0.$$

This is true when $ya = xb$. Thus, it follows that the equality condition is:

$$\boxed{\frac{x}{y} = \frac{a}{b}}.$$

Equality condition for three terms (second power). Let us prove the equality is possible (for three terms):

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = \frac{(x + y + z)^2}{a + b + c}.$$

For the equality condition we have

$$\frac{x}{a} = \frac{y}{b}.$$

In a similar way, we have that

$$\frac{x+y}{a+b} = \frac{z}{c}.$$

Denote

$$\frac{x}{a} = \frac{y}{b} = k.$$

Then

$$x = ka,$$

and

$$y = kb.$$

Plug-in to get that

$$\frac{ka + kb}{a + b} = \frac{z}{c}.$$

We obtain

$$\frac{k(a + b)}{a + b} = \frac{z}{c}.$$

Hence

$$k = \frac{z}{c}.$$

Thus, it follows that the equality condition is:

$$\boxed{\frac{x}{a} = \frac{y}{b} = \frac{z}{c}}.$$

Equality condition for n -terms (second power). In a similar way, we can get the inequality condition for 4, 5, 6, 7,... terms until n terms. Thus, it follows that the equality condition for n terms is:

$$\boxed{\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = \dots = \frac{a_n}{b_n}}.$$

Main results and proofs

Generalization for cubes. Let b_1, b_2, c_1, c_2 be positive numbers, then

$$\boxed{\frac{a_1^3}{b_1 \cdot c_1} + \frac{a_2^3}{b_2 \cdot c_2} \geq \frac{(a_1 + a_2)^3}{(b_1 + b_2)(c_1 + c_2)}}.$$

Proof. Rewrite it using the common denominator and opening the paranthesis:

$$\frac{b_2 c_2 a_1^3 + b_1 c_1 a_2^3}{b_1 b_2 c_1 c_2} \geq \frac{a_1^3 + 3a_1^2 a_2 + 3a_1 a_2^2 + a_2^3}{c_1 b_1 + c_1 b_2 + c_2 b_1 + c_2 b_2}.$$

Cross-multiply to get

$$\begin{aligned} a_1^3 b_1 c_1 b_2 c_2 + a_1^3 b_2^2 c_1 c_2 + a_1^3 c_2^2 b_1 b_2 + a_1^3 b_2^2 c_2^2 + a_2^3 b_1^2 c_1^2 + a_2^3 c_1^2 b_1 b_2 + a_2^3 b_1^2 c_1 c_2 + \\ + a_2^3 b_1 c_1 b_2 c_2 \geq a_1^3 b_1 c_1 b_2 c_2 + 3a_1^2 a_2 b_1 b_2 c_1 c_2 + 3a_1 a_2^2 b_1 b_2 c_1 c_2 + a_2^3 b_1 c_1 b_2 c_2. \end{aligned}$$

Cancel the like terms to obtain

$$a_1^3 b_2^2 c_1 c_2 + a_1^3 c_2^2 b_1 b_2 + a_1^3 b_2^2 c_2^2 + a_2^3 b_1^2 c_1^2 + a_2^3 c_1^2 b_1 b_2 + a_2^3 b_1^2 c_1 c_2 \geq 3a_1^2 a_2 b_1 b_2 c_1 c_2 + 3a_1 a_2^2 b_1 b_2 c_1 c_2.$$

We apply the inequality $A + B + C \geq 3\sqrt[3]{ABC}$ to get

$$a_1^3 b_2^2 c_1 c_2 + a_1^3 c_2^2 b_1 b_2 + a_2^3 b_1^2 c_1^2 \geq 3\sqrt[3]{a_1^3 b_2^2 c_1 c_2 \cdot a_1^3 c_2^2 b_1 b_2 \cdot a_2^3 b_1^2 c_1^2} = 3 \cdot a_1^2 a_2 b_1 b_2 c_1 c_2.$$

$$a_1^3 b_2^2 c_2^2 + a_2^3 c_1^2 b_1 b_2 + a_2^3 b_1^2 c_1 c_2 \geq 3\sqrt[3]{a_1^3 b_2^2 c_2^2 \cdot a_2^3 c_1^2 b_1 b_2 \cdot a_2^3 b_1^2 c_1 c_2} = 3 \cdot a_2^2 a_1 b_1 b_2 c_1 c_2.$$

Summing up the last two inequalities ends the proof (for cubes with two terms).

Equality condition for two terms (third power). It is known that in the inequality

$$A + B + C \geq 3\sqrt[3]{ABC},$$

the equality holds true when $A = B = C$. This means that

$$a_1^3 b_2^2 c_2^2 = a_2^3 c_1^2 b_1 b_2 = a_2^3 b_1^2 c_1 c_2,$$

and

$$a_1^3 b_2^2 c_2^2 = a_2^3 c_1^2 b_1 b_2 = a_2^3 b_1^2 c_1 c_2.$$

Now, let us prove that the equality condition for cubes with three terms is:

$$\boxed{\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}}.$$

It is known that in the inequality

$$A + B + C \geq 3\sqrt[3]{ABC},$$

the equality holds true when $A = B = C$,

$$a_1^3 b_2^2 c_1 c_2 = a_1^3 c_2^2 b_1 b_2.$$

Cancel similar terms to get

$$b_2 c_1 = c_2 b_1.$$

Divide by b_2 and c_2 to get this:

$$\frac{b_1}{b_2} = \frac{c_1}{c_2}.$$

Let us equalize these two terms to get

$$a_1^3 b_2^2 c_1 c_2 = a_1^3 c_2^2 b_1 b_2,$$

and these two also

$$a_1^3 b_2^2 c_2^2 = a_2^3 c_1^2 b_1 b_2.$$

Then, divide one equation by the other to obtain

$$\frac{a_1^3 b_2^2 c_1 c_2}{a_1^3 b_2^2 c_2^2} = \frac{a_1^3 c_2^2 b_1 b_2}{a_2^3 c_1^2 b_1 b_2}.$$

Cancel similar terms

$$\frac{c_1}{c_2} = \frac{a_1^3 c_2^2}{a_2^3 c_1^2}.$$

Since $\frac{b_1}{b_2} = \frac{c_1}{c_2}$, then let us rewrite this as

$$\frac{b_1}{b_2} = \frac{a_1^3 b_2^2}{a_2^3 b_1^2}.$$

Divide by $\frac{b_2^2}{b_1^2}$ to get

$$\frac{b_1^3}{b_2^3} = \frac{a_1^3}{a_2^3},$$

and with that we can affirm that

$$\frac{b_1}{b_2} = \frac{a_1}{a_2}.$$

Thus, it follows that the equality condition for cubes with two terms is:

$$\boxed{\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}}.$$

Now, let us prove the inequality for cubes with three terms, that is:

$$\boxed{\frac{a_1^3}{b_1 \cdot c_1} + \frac{a_2^3}{b_2 \cdot c_2} + \frac{a_3^3}{b_3 \cdot c_3} \geq \frac{(a_1 + a_2 + a_3)^3}{(b_1 + b_2 + b_3)(c_1 + c_2 + c_3)}}.$$

Proof. Using the inequality for cubes with two terms, we get

$$\frac{a_1^3}{b_1 \cdot c_1} + \frac{a_2^3}{b_2 \cdot c_2} + \frac{a_3^3}{b_3 \cdot c_3} \geq \frac{(a_1 + a_2)^3}{(b_1 + b_2)(c_1 + c_2)} + \frac{a_3^3}{b_3 \cdot c_3}.$$

Now, let us prove that

$$\frac{(a_1 + a_2)^3}{(b_1 + b_2)(c_1 + c_2)} + \frac{a_3^3}{b_3 \cdot c_3} \geq \frac{(a_1 + a_2 + a_3)^3}{(b_1 + b_2 + b_3)(c_1 + c_2 + c_3)}.$$

Denote $a_1 + a_2 = x$, $b_1 + b_2 = y$, and $c_1 + c_2 = z$, then applying the inequality for cubes with two terms we get

$$\frac{(a_1 + a_2)^3}{(b_1 + b_2)(c_1 + c_2)} + \frac{a_3^3}{b_3 \cdot c_3} = \frac{x^3}{y \cdot z} + \frac{a_3^3}{b_3 \cdot c_3} \geq \frac{(x + a_3)^3}{(y + b_3)(z + c_3)} = \frac{(a_1 + a_2 + a_3)^3}{(b_1 + b_2 + b_3)(c_1 + c_2 + c_3)}.$$

This ends the proof, as we get

$$\frac{a_1^3}{b_1 \cdot c_1} + \frac{a_2^3}{b_2 \cdot c_2} + \frac{a_3^3}{b_3 \cdot c_3} \geq \frac{(a_1 + a_2 + a_3)^3}{(b_1 + b_2 + b_3)(c_1 + c_2 + c_3)}.$$

The equality condition for cubes (with two terms) is:

$$\boxed{\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}},$$

in a similar way we need to have the following equality condition

$$\boxed{\frac{a_1 + a_2}{a_3} = \frac{b_1 + b_2}{b_3} = \frac{c_1 + c_2}{c_3}}.$$

Denote

$$\boxed{\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = k},$$

then we get

$$a_1 = ka_2, \quad b_1 = kb_2, \quad c_1 = kc_2.$$

Plug this in to get

$$\frac{ka_2 + a_2}{a_3} = \frac{kb_2 + b_2}{b_3} = \frac{kc_2 + c_2}{c_3}.$$

$$\frac{a_2(k+1)}{a_3} = \frac{b_2(k+1)}{b_3} = \frac{c_2(k+1)}{c_3}.$$

$$\frac{a_2}{a_3} = \frac{b_2}{b_3} = \frac{c_2}{c_3}.$$

So, the equality condition for cubes (with three terms) is:

$$\left\{ \begin{array}{l} \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}, \\ \frac{b_1}{c_1} = \frac{b_2}{c_2} = \frac{b_3}{c_3}. \end{array} \right.$$

We proved the inequality for cubes with two terms and the inequality for cubes with three terms. In a similar way, we can get the inequality for 4, 5, 6, 7,... terms until n terms, that is:

$$\frac{a_1^3}{b_1 \cdot c_1} + \frac{a_2^3}{b_2 \cdot c_2} + \frac{a_3^3}{b_3 \cdot c_3} + \dots + \frac{a_n^3}{b_n \cdot c_n} \geq \frac{(a_1 + a_2 + a_3 + \dots a_n)^3}{(b_1 + b_2 + b_3 + \dots + b_n)(c_1 + c_2 + c_3 + \dots + c_n)}.$$

And the equality condition for cubes with n terms is:

$$\left\{ \begin{array}{l} \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = \dots = \frac{a_n}{b_n}, \\ \frac{b_1}{c_1} = \frac{b_2}{c_2} = \frac{b_3}{c_3} = \dots = \frac{b_n}{c_n}. \end{array} \right.$$

Now, let us try to generalize this inequality for higher powers. We start with two terms. Let us prove the following generalization:

$$\frac{a_1^k}{b_1 \cdot c_1 \cdot \dots \cdot \gamma_1} + \frac{a_2^k}{b_2 \cdot c_2 \cdot \dots \cdot \gamma_2} \geq \frac{(a_1 + a_2)^k}{(b_1 + b_2)(c_1 + c_2) \dots (\gamma_1 + \gamma_2)}.$$

We proceed the proof by mathematical induction with respect to k .

Base case. If $k = 2$, then we need to prove that

$$\frac{a_1^2}{b_1 \cdot c_1} + \frac{a_2^2}{b_2 \cdot c_2} \geq \frac{(a_1 + a_2)^2}{(b_1 + b_2)(c_1 + c_2)}.$$

We already proved this.

Induction hypothesis. Assume that the statement is true for $k = m$, let us prove that the statement is true for $k = m + 1$. So, we have

$$\underbrace{\frac{a_1^m}{b_1 \cdot c_1 \cdot \dots \cdot \gamma_1}}_{m-1 \text{ terms}} + \underbrace{\frac{a_2^m}{b_2 \cdot c_2 \cdot \dots \cdot \gamma_2}}_{m-1 \text{ terms}} \geq \frac{(a_1 + a_2)^m}{(b_1 + b_2)(c_1 + c_2) \cdot \dots \cdot (\gamma_1 + \gamma_2)},$$

and we need to prove that the following is true:

$$\underbrace{\frac{a_1^{m+1}}{b_1 \cdot c_1 \cdot \dots \cdot \gamma_1 \cdot y_1}}_{m \text{ terms}} + \underbrace{\frac{a_2^{m+1}}{b_2 \cdot c_2 \cdot \dots \cdot \gamma_2 \cdot y_2}}_{m \text{ terms}} \geq \frac{(a_1 + a_2)^{m+1}}{(b_1 + b_2)(c_1 + c_2) \cdot \dots \cdot (\gamma_1 + \gamma_2)(y_1 + y_2)}.$$

In order to be able to use the induction hypothesis for $k = m$ let us rewrite this last inequality in the following way:

$$\underbrace{\frac{a_1^m}{\frac{b_1 y_1}{a_1} \cdot c_1 \cdot \dots \cdot \gamma_1}}_{m-1 \text{ terms}} + \underbrace{\frac{a_2^m}{\frac{b_2 y_2}{a_2} \cdot c_2 \cdot \dots \cdot \gamma_2}}_{m-1 \text{ terms}} \geq \frac{(a_1 + a_2)^{m+1}}{(b_1 + b_2)(c_1 + c_2) \cdot \dots \cdot (\gamma_1 + \gamma_2)(y_1 + y_2)}.$$

Let us use the induction hypothesis, since we now have a_1 to the power m , assuming that the first inequality of the induction hypothesis is true we get:

$$\underbrace{\frac{a_1^m}{b_1 \cdot c_1 \cdot \dots \cdot \gamma_1}}_{m-1 \text{ terms}} + \underbrace{\frac{a_2^m}{b_2 \cdot c_2 \cdot \dots \cdot \gamma_2}}_{m-1 \text{ terms}} \geq \frac{(a_1 + a_2)^m}{(b_1 + b_2)(c_1 + c_2) \cdot \dots \cdot (\gamma_1 + \gamma_2)},$$

where the equality condition holds, when:

$$\boxed{\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \dots = \frac{\gamma_1}{\gamma_2} = \frac{y_1}{y_2}}.$$

Using induction hypothesis, we get that:

$$\underbrace{\frac{a_1^m}{\frac{b_1 y_1}{a_1} \cdot c_1 \cdot \dots \cdot \gamma_1}}_{m-1 \text{ terms}} + \underbrace{\frac{a_2^m}{\frac{b_2 y_2}{a_2} \cdot c_2 \cdot \dots \cdot \gamma_2}}_{m-1 \text{ terms}} \geq \frac{(a_1 + a_2)^m}{(\frac{b_1 y_1}{a_1} + \frac{b_2 y_2}{a_2})(c_1 + c_2) \cdot \dots \cdot (\gamma_1 + \gamma_2)}.$$

To finish the proof for $m + 1$ power, we need to prove that

$$\frac{(a_1 + a_2)^m}{(\frac{b_1 y_1}{a_1} + \frac{b_2 y_2}{a_2})(c_1 + c_2) \cdot \dots \cdot (\gamma_1 + \gamma_2)} \geq \frac{(a_1 + a_2)^{m+1}}{(b_1 + b_2)(c_1 + c_2) \cdot \dots \cdot (\gamma_1 + \gamma_2)(y_1 + y_2)}.$$

Then we cancel like terms

$$\frac{1}{\frac{b_1 y_1}{a_1} + \frac{b_2 y_2}{a_2}} \geq \frac{(a_1 + a_2)}{(b_1 + b_2)(y_1 + y_2)}.$$

Then we cross multiply

$$b_1 y_1 + b_1 y_2 + b_2 y_1 + b_2 y_2 \geq b_1 y_1 + \frac{b_2 y_2 a_1}{a_2} + b_2 y_2 + \frac{b_1 y_1 a_2}{a_1}.$$

Then we cancel like terms

$$b_1 y_2 + b_2 y_1 \geq \frac{b_2 y_2 a_1}{a_2} + \frac{b_1 y_1 a_2}{a_1}.$$

To then rewrite the expression like this:

$$b_1 y_2 + b_2 y_1 \geq \frac{b_2 y_2 a_1^2 + b_1 y_1 a_2^2}{a_1 a_2}.$$

And we multiply with $a_1 a_2$ on both sides

$$b_1 y_2 a_1 a_2 + b_2 y_1 a_1 a_2 \geq b_2 y_2 a_1^2 + b_1 y_1 a_2^2.$$

Let us rewrite it like this:

$$b_1 y_2 a_1 a_2 + b_2 y_2 a_1^2 \geq b_2 y_1 a_1 a_2 + b_1 y_1 a_2^2.$$

To then factorize and bring all terms to the left side of the inequality

$$y_2 a_1 (b_1 a_2 - b_2 a_1) + y_1 a_2 (b_2 a_1 - b_1 a_2) \geq 0.$$

Let us rewrite like this:

$$(b_1 a_2 - b_2 a_1)(y_2 a_1 - y_1 a_2) \geq 0.$$

So we can say that:

$$b_1 a_2 - b_2 a_1 \geq 0,$$

so it follows that

$$b_1 a_2 \geq b_2 a_1,$$

and also

$$\frac{b_1}{b_2} \geq \frac{a_1}{a_2}.$$

And that

$$y_2 a_1 - y_1 a_2 \geq 0,$$

so it follows that

$$y_2 a_1 \geq y_1 a_2,$$

and also

$$\frac{y_1}{y_2} \leq \frac{a_1}{a_2}.$$

We can now affirm that

$$\frac{b_1}{b_2} \geq \frac{a_1}{a_2} \geq \frac{y_1}{y_2}.$$

By homogeneity, affirm that

$$\frac{a_1}{a_2} \geq \frac{b_1}{b_2} \geq \frac{y_1}{y_2}.$$

$$\underbrace{\frac{a_1^m}{b_1 \cdot c_1 \cdot \dots \cdot \gamma_1}}_{m-1 \text{ terms}} + \underbrace{\frac{a_2^m}{b_2 \cdot c_2 \cdot \dots \cdot \gamma_2}}_{m-1 \text{ terms}} \geq \frac{(a_1 + a_2)^m}{(b_1 + b_2)(c_1 + c_2) \cdot \dots \cdot (\gamma_1 + \gamma_2)},$$

this equation holds true when

$$\boxed{\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{y_1}{y_2}}.$$

Now with three terms

$$\underbrace{\frac{a_1^m}{b_1 \cdot c_1 \cdot \dots \cdot \gamma_1}}_{m-1 \text{ terms}} + \underbrace{\frac{a_2^m}{b_2 \cdot c_2 \cdot \dots \cdot \gamma_2}}_{m-1 \text{ terms}} + \underbrace{\frac{a_3^m}{b_3 \cdot c_3 \cdot \dots \cdot \gamma_3}}_{m-1 \text{ terms}} \geq \frac{(a_1 + a_2 + a_3)^m}{(b_1 + b_2 + b_3)(c_1 + c_2 + c_3) \cdot \dots \cdot (\gamma_1 + \gamma_2 + \gamma_3)},$$

$$\frac{a_1^m}{b_1 \cdot c_1 \cdot \dots \cdot \gamma_1} + \frac{(a_2 + a_3)^m}{(b_2 + b_3)(c_2 + c_3) \cdot \dots \cdot (\gamma_2 + \gamma_3)} \geq \frac{(a_1 + a_2 + a_3)^m}{(b_1 + b_2 + b_3)(c_1 + c_2 + c_3) \cdot \dots \cdot (\gamma_1 + \gamma_2 + \gamma_3)}.$$

Let us rewrite $a_2 + a_3$ as a'_2 , $b_2 + b_3$ as b'_2 , $c_2 + c_3$ as c'_2 , and $\gamma_2 + \gamma_3$ as γ'_2 .

$$\frac{a_1^m}{b_1 \cdot c_1 \cdot \dots \cdot \gamma_1} + \frac{(a'_2)^m}{(b'_2)(c'_2) \cdot \dots \cdot (\gamma'_2)} \geq \frac{(a_1 + a'_2)^m}{(b_1 + b'_2)(c_1 + c'_2) \cdot \dots \cdot (\gamma_1 + \gamma'_2)}.$$

We already proved this.

The equality condition is the following:

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \dots = \frac{\gamma_1}{\gamma_2},$$

so

$$\frac{a_2 + a_3}{a_1} = \frac{b_2 + b_3}{b_1} = \frac{c_2 + c_3}{c_1} = \dots = \frac{\gamma_2 + \gamma_3}{\gamma_1}.$$

Denote

$$\boxed{\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \dots = \frac{\gamma_1}{\gamma_2} = k,}$$

then

$$\frac{a_2 + a_3}{a_2 k} = \frac{b_2 + b_3}{b_2 k} = \frac{c_2 + c_3}{c_2 k} = \dots = \frac{\gamma_2 + \gamma_3}{\gamma_2 k}.$$

$$\frac{a_2}{a_2 k} + \frac{a_3}{a_2 k} = \frac{b_2}{b_2 k} + \frac{b_3}{b_2 k} = \frac{c_2}{c_2 k} + \frac{c_3}{c_2 k} = \dots = \frac{\gamma_2}{\gamma_2 k} + \frac{\gamma_3}{\gamma_2 k}.$$

Thus, it follows that

$$\boxed{\frac{a_3}{a_2} = \frac{b_3}{b_2} = \frac{c_3}{c_2} = \dots = \frac{\gamma_3}{\gamma_2}.}$$

Combining the equations in the last two boxes we get

$$\left\{ \begin{array}{l} \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}, \\ \frac{b_1}{c_1} = \frac{b_2}{c_2} = \frac{b_3}{c_3}, \\ \vdots \\ \frac{y_1}{\gamma_1} = \frac{y_2}{\gamma_2} = \frac{y_3}{\gamma_3}. \end{array} \right.$$

In the same way as 2 terms, 3 terms, we can prove this for n terms, so:

$$\boxed{\underbrace{\frac{a_1^k}{b_1 \cdot c_1 \cdot \dots \cdot \gamma_1}}_{k-1 \text{ terms}} + \underbrace{\frac{a_2^k}{b_2 \cdot c_2 \cdot \dots \cdot \gamma_2}}_{k-1 \text{ terms}} + \dots + \underbrace{\frac{a_n^k}{b_n \cdot c_n \cdot \dots \cdot \gamma_n}}_{k-1 \text{ terms}} \geq \frac{(a_1 + a_2 + \dots + a_n)^k}{(b_1 + b_2 + \dots + b_n)(c_1 + c_2 + \dots + c_n) \dots (\gamma_1 + \gamma_2 + \dots + \gamma_n)},}$$

and the equality condition for this case is:

$$\left\{ \begin{array}{l} \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = \dots = \frac{a_n}{b_n}, \\ \frac{b_1}{c_1} = \frac{b_2}{c_2} = \frac{b_3}{c_3} = \dots = \frac{b_n}{c_n}, \\ \vdots \\ \frac{y_1}{\gamma_1} = \frac{y_2}{\gamma_2} = \frac{y_3}{\gamma_3} = \dots = \frac{y_n}{\gamma_n}. \end{array} \right.$$

Applications to illustrate the usefulness

These inequalities, both for squares and their generalizations (for cubes and the n -th power), can be applied in various areas of mathematics, such as algebra, algebraic inequalities, geometry, distance geometry, knot theory, and more. We provide below few more examples:

Mathematical Olympiads and Problem Solving (inequality manipulation and estimation, symmetric inequality proofs, cyclic sum problems, bounding expressions with constraints, proving extremal values (min/max)).

Number Theory (bounding arithmetic functions, estimating divisor sums, working with sequences and series, diophantine inequality constraints).

Discrete Mathematics (analysis of sums over finite sets, graph theory bounds (e.g., degree sums), combinatorial inequalities, probabilistic method estimates).

Mathematical Analysis (bounding integrals, series convergence estimates, norm inequalities, functional inequalities).

Optimization Theory (constrained optimization problems, Lagrange multiplier estimates, convexity-based inequalities, bounding objective functions).

Linear Algebra (vector norm bounds, inner product inequalities, matrix inequality estimates, quadratic form bounds).

Probability and Statistics (variance and expectation bounds, moment inequalities, estimating statistical error, concentration inequalities).

Computer Science (algorithm complexity bounds, approximation algorithms, competitive analysis, information theory estimates).

Mathematical Physics (energy estimates, variational methods, stability bounds, quantum state norm estimates).

Application 1 (National Math Competition, Grade 11-12, Arm 2026).

Let n be a positive integer and a, b, c be real numbers, such that $abc = 1$. Prove that

$$\frac{1+a^n}{1+b} + \frac{1+b^n}{1+c} + \frac{1+c^n}{1+a} \geq 3$$

Proof of the application. Let us rewrite the inequality like this and apply both Sedrakyan's inequality for squares and the generalization for n^{th} power:

$$\frac{1}{1+b} + \frac{a^n}{1+b} + \frac{1}{1+c} + \frac{b^n}{1+c} + \frac{1}{1+a} + \frac{c^n}{1+a} \geq \frac{(1+1+1)^2}{3+a+b+c} + \frac{(a+b+c)^n}{(1+1+1)^{n-2}(3+a+b+c)}.$$

Note that the following inequality is true using inequality for squares:

$$\frac{1}{1+b} + \frac{1}{1+c} + \frac{1}{1+a} \geq \frac{(1+1+1)^2}{3+a+b+c},$$

Now, let us prove that

$$\frac{a^n}{1+b} + \frac{b^n}{1+c} + \frac{c^n}{1+a} \geq \frac{(a+b+c)^n}{(1+1+1)^{n-2}(3+a+b+c)}.$$

In order to be able to apply the inequality with n -th power, we take $b_1 = 1+b$, $b_2 = 1+c$, $b_3 = 1+a$, and $c_1 = 1$, $c_2 = 1$, $c_3 = 1$, and $\gamma_1 = 1$, $\gamma_2 = 1$, $\gamma_3 = 1$, then we get

$$\frac{a^n}{b_1 \cdot \underbrace{1 \cdot \dots \cdot 1}_{n-2 \text{ terms}}} + \frac{b^n}{b_2 \cdot \underbrace{1 \cdot \dots \cdot 1}_{n-2 \text{ terms}}} + \frac{c^n}{b_3 \cdot \underbrace{1 \cdot \dots \cdot 1}_{n-2 \text{ terms}}} \geq \frac{(a+b+c)^n}{(b_1+b_2+b_3) \underbrace{(1+1+1) \cdot \dots \cdot (1+1+1)}_{n-2 \text{ terms}}}.$$

Plugging in the values of $b_1 = 1+b$, $b_2 = 1+c$, $b_3 = 1+a$, we get

$$\frac{a^n}{1+b} + \frac{b^n}{1+c} + \frac{c^n}{1+a} \geq \frac{(a+b+c)^n}{(1+1+1)^{n-2}(3+a+b+c)}.$$

In order to finish the proof, we need to prove that

$$\frac{(1+1+1)^2}{3+a+b+c} + \frac{(a+b+c)^n}{(1+1+1)^{n-2}(3+a+b+c)} \geq 3.$$

Then with the common denominator we can addition to get

$$\frac{(1+1+1)^2(1+1+1)^{n-2} + (a+b+c)^n}{(3+a+b+c)(1+1+1)^{n-2}} \geq 3,$$

And we rewrite as

$$\frac{(1+1+1)^n + (a+b+c)^n}{(3+a+b+c)(1+1+1)^{n-2}} \geq 3.$$

Cross-multiply to get

$$3^2 \cdot 3^{n-2} + (a+b+c)^n \geq 3 \cdot 3^{n-2}(3+a+b+c).$$

Denote by $a+b+c = x$, then the last inequality is equivalent to proving that

$$3^n + x^n \geq 3^n + 3^{n-1}x.$$

So, we need to prove that:

$$x^{n-1} \geq 3^{n-1}, \text{ that is } x \geq 3.$$

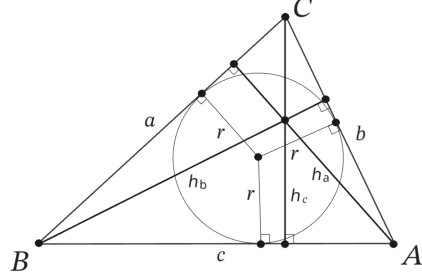
Or equivalently, we need to prove that $a+b+c \geq 3$. Given that $abc = 1$, from AM-GM we get

$$A^3 + B^3 + C^3 \geq 3ABC, \text{ therefore } a+b+c \geq 3\sqrt[3]{abc} = 3.$$

Application 2. Let ABC be a triangle with sides a, b, c (see figure). Let h_a, h_b, h_c be the corresponding altitudes. Prove that

$$\frac{a}{h_a} + \frac{b}{h_b} + \frac{c}{h_c} \geq \frac{a+b+c}{3r},$$

where r is the inradius.



Solution. We need to prove that

$$\frac{a}{h_a} + \frac{b}{h_b} + \frac{c}{h_c} \geq \frac{a+b+c}{3r}.$$

According to Sedrakyan's inequality (Titu's lemma), we have

$$\frac{a^2}{ah_a} + \frac{b^2}{bh_b} + \frac{c^2}{ch_c} \geq \frac{(a+b+c)^2}{ah_a + bh_b + ch_c}.$$

So, to prove the initial inequality it is enough to prove that

$$\frac{(a+b+c)^2}{ah_a + bh_b + ch_c} \geq \frac{a+b+c}{3r}.$$

Canceling $a+b+c$ on both sides, we get

$$\frac{ah_a + bh_b + ch_c}{3} \leq (a+b+c)r.$$

Divide both hand-sides by 2 to get

$$\frac{ah_a + bh_b + ch_c}{6} \leq \frac{(a+b+c)r}{2}.$$

Let S be the area of the triangle, then let us use the following well-known formulas:

$$S = \frac{ah_a}{2} = \frac{bh_b}{2} = \frac{ch_c}{2} = \frac{(a+b+c)r}{2}.$$

Thus, it follows that

$$\frac{ah_a + bh_b + ch_c}{6} \leq S,$$

We need to prove $\frac{3S}{3} \leq S$, It is equivalent to $S \leq S$. This ends the proof.

Application 3 (knot theory). This inequality can be used in knot-theory as well. The *rope length* of a knot is the minimum length required to tie it.

Example estimate structure, by applying Sedrakyan's inequality (Titu's lemma) we get the following:

$$\Sigma \frac{k_i^2}{l_i} \geq \frac{(\Sigma k_i)^2}{\Sigma l_i}$$

where

- k_i : curvature contributions

- l_i : arc lengths

Application 4 (Bounds on sums of squared distances). In distance geometry we often analyze expressions like

$$\sum_{i=1}^n \frac{d_i^2}{w_i}$$

where

- d_i = distances

- w_i = positive weights (edge lengths, uncertainties, or masses).

By applying Sedrakyan's inequality (Titu's lemma) we get the following:

$$\Sigma \frac{d_i^2}{w_i} \geq \frac{(\Sigma d_i)^2}{\Sigma w_i}.$$

This gives lower bounds for total distance-based energies.

Application 5 (Sensor network localization).

In many problems of distance-based positioning, we know distances from sensors to anchors.

Typical least-squares objective:

$$\Sigma \frac{(d_i - \hat{d}_i)^2}{w_i}$$

By applying Sedrakyan's inequality (Titu's lemma) we get the following:

$$\Sigma \frac{(d_i - \hat{d}_i)^2}{w_i} \geq \frac{(\Sigma (d_i - \hat{d}_i))^2}{\Sigma w_i}$$

- error propagation
- stability of reconstruction
- robustness of localization algorithms.

Application 6 (Stress functions in multidimensional scaling)

In Multidimensional Scaling (MDS), we reconstruct coordinates from pairwise distances. The objective often looks like

$$\sum_{i < j} \frac{(d_{ij} - \delta_{ij})^2}{w_{ij}}$$

By applying Sedrakyan's inequality (Titu's lemma) we get the following:

$$\sum_{i < j} \frac{(d_{ij} - \delta_{ij})^2}{w_{ij}} \geq \frac{(\sum_{i < j} (d_{ij} - \delta_{ij}))^2}{\sum_{i < j} w_{ij}}$$

- d_{ij} = model distance
- δ_{ij} = observed distance.

Application 7 (Where the generalized n-power form could appear)

The power version

$$\sum \frac{a_i^n}{b_i^{n-1}}$$

By applying the inequality for n -th power we get

$$\sum \frac{a_i^n}{b_i^{n-1}} \geq \frac{(\sum a_i)^n}{\sum b_i^{n-1}}$$

could theoretically appear when bounding:

- distance energies
- graph embeddings
- network stress functions

These show up in:

- distance geometry optimization
- spectral graph theory
- geometric analysis.

Application 8 (Polygonal knot energy estimates).

In geometric knot theory (a metric side of knot theory), knots are often approximated by polygons with edges e_i .

Many knot energies depend on pairwise distances between segments:

$$E = \sum_{i \neq j} \frac{l_i^n}{d_{ij}^{n-1}}.$$

By applying the inequality for n -th power we get

$$E = \sum_{i \neq j} \frac{l_i^n}{d_{ij}^{n-1}} \geq \frac{(\sum l_i)^n}{\sum_{i \neq j} d_{ij}^{n-1}}.$$

Application 9 (Discrete distance-energy bounds (distance geometry))

In distance geometry, one studies configurations of points $p_1, \dots, p_m$ with pairwise distances d_{ij} . Many optimization problems involve distance energies, for example

$$E = \sum_{i < j} \frac{d_{ij}^n}{w_{ij}^{n-1}}$$

where $w_{ij} > 0$ are weights

Applying the n-power Sedrakyan form:

$$E \geq \frac{(\sum_{i < j} d_{ij})^n}{(\sum_{i < j} w_{ij})^{n-1}}$$

This gives a global lower bound on total distance energy based only on the sums of distances and weights.

Such bounds appear when studying:

- rigidity problems
- embedding distortion
- weighted distance matrices.

Application 10 (Bounding stress functions in graph embeddings)

In geometric embedding problems (a topic connected to distance geometry), one minimizes a stress functional

$$S = \sum_{i < j} \frac{(d_{ij} - \delta_{ij})^n}{w_{ij}^{n-1}}$$

where

- d_{ij} = realized distances
- δ_{ij} = target distances.

Using the generalized Sedrakyan inequality:

$$S \geq \frac{(\sum |d_{ij} - \delta_{ij}|)^n}{(\sum w_{ij})^{n-1}}$$

This gives a lower bound on total embedding error.

These ideas appear in:

- multidimensional scaling
- network embeddings
- rigidity optimization.

References

- [1] N. Sedrakyan, *About the applications of one useful inequality*, Kvant Journal, vol. 97, no. 2, pp. 42–44, 1997.
- [2] H. Sedrakyan and N. Sedrakyan, *Algebraic Inequalities*, Problem Books in Mathematics, Springer International Publishing, 2018. doi: 10.1007/978-3-319-77836-5
- [3] A. Engel, *Problem-Solving Strategies*, Problem Books in Mathematics, Springer, 1998. doi: 10.1007/b97682.
- [4] T. Andreescu and B. Enescu, *Mathematical Olympiad Treasures*, 2nd ed., Birkhäuser Boston, 2007. doi: 10.1007/978-0-8176-8253-8.
- [5] H. Sedrakyan and N. Sedrakyan, *Geometric Inequalities: Methods of Proving*, Problem Books in Mathematics, Springer International Publishing, 2017. doi: 10.1007/978-3-319-55080-0.